RQM Spins, Spinors, and Boosts

Robert D. Klauber September 30, 2016
www.quantumfieldtheory.info

1 The spinor basis vectors

The basis vectors for 4D spin (ignoring antiparticle v_2 and v_1) are

$$u_1 = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 \\ 0 \\ \frac{p^3}{E+m} \\ \frac{p^1+ip^2}{E+m} \end{pmatrix} \quad u_2 = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 0 \\ 1 \\ \frac{p^1-ip^2}{E+m} \\ \frac{-p^3}{E+m} \end{pmatrix} \quad \text{from (4-20), pg. 89 of Klauber.} \tag{1}$$

2 General state of fermion (with spin ½)

$$u = C_1(\mathbf{p})u_1(\mathbf{p}) + C_2(\mathbf{p})u_2(\mathbf{p}) \tag{2}$$

3 Spinors: Unboosted and boosted

3.1 Unboosted spinor

The unboosted ($\mathbf{p} = 0$) spinors u_1 and u_2 are

$$u_1 = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad u_2 = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad (\text{actually } E = m \text{ as well, for } \mathbf{p} = 0) \tag{3}$$

3.2 Boosted spinor

A boosted spinor $u_1(\mathbf{p} \neq 0)$ or $u_2(\mathbf{p} \neq 0)$ is represented in (1)

4 General state of fermion: Unboosted and boosted

4.1 Unboosted general state of a fermion with spin up

An electron at rest in a spin up state (spin in plus x^3 direction) is represented by[1]

$$u(\mathbf{p}=0) = u_1(\mathbf{p}=0) \qquad (C_1 = 1;\ C_2 = 0 \text{ in (2)}) \tag{4}$$

4.2 Boosted general state of above fermion to $v^1 \approx c$, $v^2 = v^3 = 0$

From (2) and (1), for the electron boosted in the x^1 direction,

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[1] Strictly speaking, we can never have an electron in an exactly perfect up state. (In the real world, spin will be off by some miniscule fraction of a degree from the x^3 axis.) So $C_2 \approx 0$, but not exactly = 0. If the classical disk of Box 4-2, pg 95, of Klauber were perfectly straight up at $\mathbf{v} = 0$, $\mathbf{L}$ would never waver from straight up, for any $\mathbf{v}$ in the x^1 direction.

$$u = C_1(\mathbf{p})u_1(\mathbf{p}) + C_2(\mathbf{p})u_2(\mathbf{p}) = C_1(\mathbf{p})\sqrt{\frac{E+m}{2m}}\begin{pmatrix}1\\0\\0\\\dfrac{p^1}{E+m}\end{pmatrix} + C_2(\mathbf{p})\sqrt{\frac{E+m}{2m}}\begin{pmatrix}0\\1\\\dfrac{p^1}{E+m}\\0\end{pmatrix}. \tag{5}$$

For the speed in that direction close to the speed of light, we have $p^1 \to E \to \infty$, so(5) becomes

$$u \approx C_1(\mathbf{p})\sqrt{\frac{E+m}{2m}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} + C_2(\mathbf{p})\sqrt{\frac{E+m}{2m}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix} = \sqrt{\frac{E+m}{2m}}\begin{pmatrix}C_1(\mathbf{p})\\C_2(\mathbf{p})\\C_2(\mathbf{p})\\C_1(\mathbf{p})\end{pmatrix} \qquad \mathbf{p} = p^1\mathbf{i}_1 \text{ with } v^1 \to c / \tag{6}$$

We know physically (see Klauber Fig. 4-1, pg 96) that (5) must have spin aligned with the x^1 axis, and thus must be an eigenstate of the spin operator (see Klauber, (4-39), pg. 93)

$$\Sigma_1 = \frac{\hbar}{2}\begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}. \tag{7}$$

That is, we need

$$\Sigma_1 u = \frac{\hbar}{2}u \tag{8}$$

or

$$\frac{\hbar}{2}\sqrt{\frac{E+m}{2m}}\begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}\begin{pmatrix}C_1(\mathbf{p})\\C_2(\mathbf{p})\\C_2(\mathbf{p})\\C_1(\mathbf{p})\end{pmatrix} = \frac{\hbar}{2}\sqrt{\frac{E+m}{2m}}\begin{pmatrix}C_2(\mathbf{p})\\C_1(\mathbf{p})\\C_1(\mathbf{p})\\C_2(\mathbf{p})\end{pmatrix} \overset{?}{=} \frac{\hbar}{2}u\ . \tag{9}$$

The last equal sign with the question mark over it is only a true equal sign if $C_1(\mathbf{p}) = C_2(\mathbf{p}) = C$ (where we introduce the symbol C). That is the only way u can represent spin in the x^1 direction. So, for this special case, we have

$$u = C_1 u_1(\mathbf{p}) + C_2 u_2(\mathbf{p}) = C\sqrt{\frac{E+m}{2m}}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\sqrt{\frac{E+m}{2m}}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \qquad C = \frac{1}{\sqrt{2}} \text{ is normalization factor; } \mathbf{p} = p^1\mathbf{i}_1\ p^1 \to \infty\,. \tag{10}$$

4.3 Comparing unboosted and boosted general spin state

Comparing (4) to (10), we see that in addition to the basis spinors changing with **p**, the normalization constants for a particle boosted by **p** ($\mathbf{v} = \mathbf{p}/m$), C_1 and C_2, change as well.

Conclusion #1. We cannot take an unboosted general spin state, such as the up state at rest of (4), and find the boosted general spin state just by inputting the new **p** into the **p** = 0 relations for $u_1(\mathbf{p})$ and $u_2(\mathbf{p})$. The spinor basis states do not, on their own, take into account the full effect of the boost.

Conclusion #2. The relationship for the general spin state of a spinor has form

$$u = C_1(\mathbf{p})u_1(\mathbf{p}) + C_2(\mathbf{p})u_2(\mathbf{p}), \tag{11}$$

with the normalization constants, as well as the basis spinors, having dependence on **p**.

5 Finding the boosted general case spin state (finding C_1(p) and C_2(p))

5.1 Representing the boost in spinor space

In the example of Sect. 4, we knew from physical principles that the boosted spin state had to be an eigenstate of Σ_1, and that enabled us to determine C_1 and C_2 from (8) and (9). But what do we do in the more general case, where the boosted **p** value could be anything (and thus we don't know so specifically what the boosted general spin state should be)?

The answer is that we have to do the spinor space equivalent of a Lorentz transformation on the unboosted general spin state u. This is discussed in Klauber, Sect. 6.3.2, pg. 171. The 4D spinor space is not the same space as the 4D spacetime of the Lorentz transformation. But they are related. A change in orientation of spin in spacetime means a correlated change in the form of the general spin state u in spinor space.

There is a transformation in spinor space, an action of a particular matrix in spinor space on u, that corresponds to a change in velocity in spacetime. This transformation, which we do not derive here or in Klauber, is called a *representation of the Lorentz group*, and is often labeled D. D depends on **p** in a way that makes everything work out correctly in spinor space for a given boost **p**.

The transformation D is shown in Klauber's (6-22), pg. 171. (References are supplied where it is derived.) It is

$$D = e^{-i(\mathbf{L}\cdot\boldsymbol{\Theta}+\mathbf{M}\cdot\mathbf{Q})} \qquad L^k = -\tfrac{i}{2}\varepsilon_{ij}{}^{k}\gamma^i\gamma^j,\ \ \Theta^k = \left(\theta^1,\theta^2,\theta^3\right),\ \ M^k = \tfrac{i}{2}\gamma^0\gamma^k,\ \ Q^k = \left(v^1,v^2,v^3\right) \tag{12}$$

where the θ_i are rotations of the observer's frame and v_i are velocities in the three coordinate directions. As **L** and **M** are 4X4 matrices in spinor space, then so is D. Thus, the general relation for finding a boosted general spin state from an unboosted one is

$$u(\mathbf{p}) = D\,u(\mathbf{p}=0), \tag{13}$$

5.2 Finding C_1(p) and C_2(p) for general boosted spin state

The steps for determining $C_1(\mathbf{p})$ and $C_2(\mathbf{p})$ for a general spin state for a particle boosted into that state whose unboosted general spin state is known follow.

1. Start with the unboosted general spin state $u(\mathbf{p} = 0)$ in spinor space.

$$u(\mathbf{p}=0) \;=\; C_1(\mathbf{p}=0)u_1(\mathbf{p}=0) \;+\; C_2(\mathbf{p}=0)u_2(\mathbf{p}=0) \tag{14}$$

2. Apply D, the Lorentz group representation of the boost (12) to (14). (As in (13).) To get $u(\mathbf{p})$.
3. Determine $u_1(\mathbf{p})$ and $u_2(\mathbf{p})$ from equation (1).
4. Use the results from steps 2 and 3, along with normalization condition (see Klauber, (4-25), pg. 90 and (4-35) to (4-37), pg. 92) to solve for boosted $C_1(\mathbf{p})$ and $C_2(\mathbf{p})$.

$$\underbrace{u(\mathbf{p})}_{\substack{\text{known}\\ \text{from \#2}}} \;=\; C_1(\mathbf{p})\underbrace{u_1(\mathbf{p})}_{\substack{\text{known}\\ \text{from \#3}}} \;+\; C_2(\mathbf{p})\underbrace{u_2(\mathbf{p})}_{\substack{\text{known}\\ \text{from \#3}}} \tag{15}$$

$$\underbrace{\left|C_1(\mathbf{p})\right|^2+\left|C_2(\mathbf{p})\right|^2=1}_{\text{from normalization}}\,.^2 \qquad (16)$$

(15) and (16) are two equations in two unknowns, $C_1(\mathbf{p})$ and $C_2(\mathbf{p})$.

[2] For Dirac particles, (see Klauber, pg.